On exit time of stable processes

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Abstract

We study the exit time $\tau = \tau_{(0,\infty)}$ for 1-dimensional strictly stable processes and express its Laplace transform at t^α as the Laplace transform of a positive random variable with explicit density. Consequently, τ satisfies some multiplicative convolution relations. For some stable processes, e.g. for the symmetric $\frac{2}{3}$-stable process, explicit formulas for the Laplace transform and the density of τ are obtained as an application.

1 Introduction

Let $\alpha \in (0,2)$ and $(X_t, \mathbb{P}^x)$ be a strictly α-stable process in $\mathbb{R}$ with characteristic function

$$\mathbb{E}^0 e^{iX_t z} = \exp\left[-t|z|^\alpha \left(1 - i\beta \tan\frac{\pi\alpha}{2} \operatorname{sgn} z\right)\right],$$

where $\beta \in [-1,1]$ and $\beta = 0$ for $\alpha = 1$. For any $D \subset \mathbb{R}$ let

$$\tau_D = \inf\{t \geq 0 \colon X_t \notin D\}$$

be the first exit time from D of the process X_t. Throughout this article we shall consider the starting point $x > 0$ and

$$\tau = \tau_{(0,\infty)},$$

the exit time of X_t from the positive half-line.

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The question of the first exit time from domains are basic for all stochastic processes. Surprisingly few exact formulas are known for stable processes. The only exceptions are Brownian motion, completely asymmetric stable processes with $\alpha > 1$ (see [4], [14], [17]) and symmetric Cauchy process ([5], see also [11]). The quotient $\hat{\tau}/\tau$ was studied for independent τ and dual $\hat{\tau}$ in [6].

Some recent results on this problem in the completely asymmetric case were obtained by T. Simon in [17] and next were applied in [15] and [18]. On the other hand, some formulas were found by A. Kuznetsov in [12], however the final expressions are complicated. M. Kwaśnicki in [13] gives an integral representation of the density of τ in the case of symmetric stable processes ($\beta = 0$).

In this article we study the exit time $\tau = \tau_{(0,\infty)}$ for 1-dimensional stable processes and give in Theorem 3 a new formula for its Laplace transform. It follows that τ satisfies some multiplicative convolution relations(Corollary 6); in particular for $\alpha > 1$ the exit time τ is the multiplicative convolution of a $1/\alpha$-stable subordinator with an explicitly given random variable $M_{\alpha,\rho}$. We generalize in this way the result of [17] for all stable processes. Applications of Theorem 3 are next given in the final part of the article. New explicit formulas for the Laplace transform and the density of τ are proven for the processes dual to those of the Doney's class $C_{1,1}$, in particular for the symmetric $\frac{2}{3}$-stable process (Proposition 7 and Corollary 8). Further applications of Theorem 3 will be presented in a forthcoming paper.

The main tool to prove the results of this article is a series representation that we obtained in [9] for the logarithm of the bivariate Laplace exponent $\kappa(\eta, \theta)$ of the ascending ladder process built from the process X_t. This application of the series representation of $\ln \kappa$ was announced in [9]. It allows to determine explicitly in Proposition 1 the inverse Stieltjes transform of the function $1/\kappa(1, \theta)$.

2 Stieltjes transform and Wiener-Hopf factors

In this part of the article we will exploit our series representation of $\kappa(1, \theta)$ from [9] by inverting a Stieltjes transform.

Recall that if μ is a positive Borel measure on $[0, \infty)$ then for any $x \in$

$(0,\infty)$ the Stieltjes transform of μ is defined by

$$\mathcal{S}\mu(\theta) = \int_0^\infty \frac{1}{\theta + x} d\mu(x) \tag{1}$$

whenever the integral converges. According to [3], a function G on $(0,\infty)$ is of the form $G(\theta) = a + \mathcal{S}\mu(\theta)$ for a positive measure μ and $a \geq 0$ if and only if

(S1) G extends to a holomorphic function in the cut plane $\mathbb{C} \setminus \mathbb{R}_-$
(S2) $G(\theta) \geq 0$ for $\theta > 0$
(S3) $\mathrm{Im}G(z) \leq 0$ for $\mathrm{Im}z > 0$.

Then the inverse Stieltjes transform is

$$\mathcal{S}^{-1}(G)(x) = -(1/\pi) \lim_{y\to 0^+} \mathrm{Im}G(-x+iy)\,, \qquad x > 0\,,$$

where the limit, in general, is in the vague sense and equals μ. If μ is absolutely continuous with a continuous density, the limit is equal to the density of μ for all $x > 0$ ([19]).

Let $\alpha \in (0,2)$ and $(X_t, \mathbb{P}^x)$ be a strictly α-stable process in $\mathbb{R}$. By $\kappa_{\alpha,\rho}(\eta,\theta)$ we denote the bivariate Laplace exponent of the ascending ladder process built from X_t. We normalize it requiring that $\kappa_{\alpha,\rho}(1,0) = 1$. To simplify the notation we will write $\kappa(\eta,\theta)$ for a fixed pair α, ρ (or equivalently a fixed process X_t). By $\hat{\kappa}$ we denote the Laplace exponent for the dual process $\hat{X}_t = -X_t$. As usually we write the positivity coefficient

$$\rho = \mathbb{P}^0(X_t \geq 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan\frac{\pi\alpha}{2}\right).$$

Proposition 1. *For $\rho \in (0,1] \setminus \{1/\alpha\}$ we have*

$$\frac{\sin(\rho\alpha\pi)}{\pi} \int_0^\infty \frac{1}{x+\theta} \frac{x^\alpha \widehat{\kappa}(1,x)}{x^{2\alpha} + 2x^\alpha \cos(\rho\alpha\pi) + 1}\, dx = \frac{1}{\kappa(1,\theta)}. \tag{2}$$

Proof. Denote $G(\theta) = 1/\kappa(1,\theta)$. The function $G(\theta)$ extends to a holomorphic function $h_1(z)$ on $\mathbb{C} \setminus \mathbb{R}_-$ (see [8],(i) p.205). Let $\mathcal{L}$ be the set of Liouville numbers. For $\theta \in (0,1)$ and $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ we have by [9]

$$G(\theta) = \exp\left(-\sum_{m=1}^\infty \frac{(-1)^{m+1}\theta^m \sin(\rho m\pi)}{m \sin(\frac{m\pi}{\alpha})} - \sum_{k=1}^\infty \frac{(-1)^{k+1}\theta^{\alpha k} \sin(\rho\alpha k\pi)}{k \sin(\alpha k\pi)}\right).$$

The right hand side of the last formula may be extended to a holomorphic function $h_2(z)$ on $\{z \in \mathbb{C}\colon |z| < 1\} \setminus \mathbb{R}_-$ defining $w^\alpha = \exp(\alpha \mathrm{Log} w)$ where $\mathrm{Log} w = \ln|w| + i\mathrm{Arg} w$, $\mathrm{Arg} w \in (-\pi, \pi]$, is the principal value of the complex logarithm. We note that $h_1 = h_2$ on $(0,1)$, hence $h_1 = h_2$ on $\{z \in \mathbb{C}\colon |z| < 1\} \setminus \mathbb{R}_-$ and h_2 extends to a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_-$, equal for $|z| > 1$ to the holomorphic extension of $\frac{1}{\kappa(1,\theta)}$ for $\theta > 1$.

In the first part of the proof we will compute

$$l(x) = -\frac{1}{\pi} \lim_{y \to 0^+} \mathrm{Im} G(-x + iy)$$

for positive x. Denote by $h(z)$ the expression under exponential of h_2. Let us compute for $0 < x < 1$

$$l(x) = -\frac{1}{\pi} \lim_{y \to 0^+} \mathrm{Im} \exp(h(-x + iy)) = -\frac{1}{\pi} e^{\mathrm{Re}(w)} \sin(\mathrm{Im}(w)),$$

where

$$w = -\sum_{m=1}^{\infty} \frac{(-1)^{m+1}(-x)^m \sin(\rho m \pi)}{m \sin(\frac{m\pi}{\alpha})} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{i\alpha k\pi} x^{\alpha k} \sin(\rho \alpha k \pi)}{k \sin(\alpha k \pi)}.$$

The last limit is justified by a standard estimation argument, that implies that in a converging power series one can enter the limit under the series. Moreover, the same argument shows that when $0 < x < 1$, we have

$$l(x) = -\frac{1}{\pi} \lim_{w \to -x, \mathrm{Im} w > 0} \mathrm{Im} G(w) = -\frac{1}{\pi} e^{\mathrm{Re}(w)} \sin(\mathrm{Im}(w)). \qquad (3)$$

Now we evaluate

$$\mathrm{Re}(w) = \sum_{m=1}^{\infty} \frac{x^m \sin(\rho m \pi)}{m \sin(\frac{m\pi}{\alpha})} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(\alpha k \pi) x^{\alpha k} \sin(\rho \alpha k \pi)}{k \sin(\alpha k \pi)},$$

$$\mathrm{Im}(w) = -\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin(\alpha k \pi) x^{\alpha k} \sin(\rho \alpha k \pi)}{k \sin(\alpha k \pi)} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{\alpha k} \sin(\rho \alpha k \pi)}{k}.$$

We will need the following formulas from [10]

$$\sum_{k=1}^{\infty} \frac{p^k \sin(k\varphi)}{k} = \arctan \frac{p \sin\varphi}{1 - p\cos\varphi}, \qquad \varphi \in (0, 2\pi),\ p^2 \le 1. \qquad (4)$$

$$\sum_{k=1}^{\infty}\frac{p^k\cos(k\varphi)}{k}=-\frac{1}{2}\log(1-2p\cos\varphi+p^2)\,,\qquad \varphi\in(0,2\pi),\ p^2\le 1. \tag{5}$$

Therefore applying a formula $\sin(\arctan u)=\frac{u}{\sqrt{1+u^2}}$ we get

$$\begin{aligned}\sin(\mathrm{Im}(w)) &= \sin(\arctan\frac{-x^\alpha\sin(\rho\alpha\pi)}{1+x^\alpha\cos(\rho\alpha\pi)})\\ &= \frac{-\frac{x^\alpha\sin(\rho\alpha\pi)}{1+x^\alpha\cos(\rho\alpha\pi)}}{\sqrt{1+\left(\frac{x^\alpha\sin(\rho\alpha\pi)}{1+x^\alpha\cos(\rho\alpha\pi)}\right)^2}}=\frac{-x^\alpha\sin(\rho\alpha\pi)}{\sqrt{x^{2\alpha}+2x^\alpha\cos(\rho\alpha\pi)+1}}.\end{aligned}$$

Now we compute $\mathrm{Re}(w)$. By (5) we get

$$\begin{aligned}\mathrm{Re}(w) &= \sum_{m=1}^{\infty}\frac{(-1)^{m+1}x^m\sin((1-\rho)m\pi)}{m\sin(\frac{m\pi}{\alpha})}\\ &\quad+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{\alpha k}\sin((1-\rho)\alpha k\pi)}{k\sin(\alpha k\pi)}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{\alpha k}\cos(\rho\alpha k\pi)}{k}\\ &= \log\widehat{\kappa}(1,x)-\frac{1}{2}\log(1+2x^\alpha\cos(\rho\alpha\pi)+x^{2\alpha}).\end{aligned}$$

Hence

$$-\frac{1}{\pi}e^{\mathrm{Re}(w)}\sin(\mathrm{Im}(w))=\frac{\sin(\rho\alpha\pi)}{\pi}\frac{x^\alpha\widehat{\kappa}(1,x)}{x^{2\alpha}+2x^\alpha\cos(\rho\alpha\pi)+1}=l(x)>0.$$

By [9, Lemma 5] we have for $\theta>1$

$$\kappa(1,\theta)=\theta^{\alpha\rho}\kappa(1,1/\theta)$$

and we use the same method and (3) to obtain

$$l(x)=-\frac{1}{\pi}\lim_{w\to -x,\mathrm{Im}w>0}\mathrm{Im}G(w)=\frac{\sin(\rho\alpha\pi)}{\pi}\frac{x^\alpha\widehat{\kappa}(1,x)}{x^{2\alpha}+2x^\alpha\cos(\rho\alpha\pi)+1} \tag{6}$$

for $x>0, x\neq 1$.

As the function $l(x)$ is continuous at $x=1$ and by [8, p.205] the limit $\lim_{w\to -1,\mathrm{Im}w>0}\mathrm{Im}G(w)$ exists, it follows that the convergence in (6) holds also for $x=1$.

Let us now justify the fact that the function $G(\theta) = 1/\kappa(1, \theta)$ is a Stieltjes transform of a positive measure μ on $\mathbb{R}^+$. We will check the conditions (S1-3) given in the beginning of this section.

The function $G(\theta)$ is strictly positive for $\theta \in (0, \infty)$ and it extends to a holomorphic function $h_1(z)$ on $\mathbb{C} \setminus \mathbb{R}_-$. Thus the conditions (S1) and (S2) are verified. In order to justify (S3), we use the following property that we proved above: the harmonic function $-(1/\pi)\mathrm{Im}G$ extends continuously to the closed upper half-space $\{\mathrm{Im} z \geq 0\}$ and its boundary values on $\mathbb{R}$ are $l(-x) > 0$ when $x < 0$ and 0 for $x \geq 0$. Taking into account the fact that $\lim_{|z|\to\infty} G(z) = 0$ ([8, p.205]), the maximum principle([2, 1.10]) implies that $\mathrm{Im}G(z) \leq 0$ on $\{\mathrm{Im} z > 0\}$ and (S3) also holds.

It follows that for a certain $a \geq 0$ we have $G(\theta) = a + \mathcal{S}(l)(\theta)$. Considering $\theta \to \infty$ we determine $a = 0$.

Finally consider any $\alpha \in (0, 2]$. Since the Lebesgue measure of the set $\mathcal{L} \cup \mathbb{Q}$ is 0 we can take a sequence α_n tending to α. Passing to the limit we obtain (2) for all $\alpha \in (0, 2]$. □

Remark. Other proofs of the fact that $1/\kappa(1, \theta)$ is the Stieltjes transform of a positive measure μ seem possible, using properties of Bernstein functions ([16]).

We deduce immediately from Proposition 1 the following corollary.

Corollary 2. *For $\rho \in [0, 1) \setminus \{1 - 1/\alpha\}$ we have*

$$\frac{\sin((1-\rho)\alpha\pi)}{\pi} \int_0^\infty \frac{1}{x+\theta} \frac{x^\alpha \kappa(1,x)}{x^{2\alpha} + 2x^\alpha \cos((1-\rho)\alpha\pi) + 1} \, dx = \frac{1}{\widehat{\kappa}(1,\theta)}. \tag{7}$$

3 Laplace transform of τ and applications

The following theorem is the main result of the article.

Theorem 3. *Let X_t be a non-spectrally positive strictly α-stable process on $\mathbb{R}$. For any $t > 0$ we have*

$$\mathbb{E}^1 e^{-t\tau} = \frac{\sin((1-\rho)\alpha\pi)}{\pi} \int_0^\infty e^{-t^{1/\alpha}x} \frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha} + 2x^\alpha \cos((1-\rho)\alpha\pi) + 1} \, dx. \tag{8}$$

Remark. Observe that the only case excluded from the Theorem 3 is well known: when X_t is a spectrally positive α-stable process starting from $X_0 = x$, $1 < \alpha < 2$, then $(\tau^x_{(0,\infty)})_{x>0}$ is a $1/\alpha$-stable subordinator and $\mathbb{E}^1 e^{-t\tau} = e^{-t^{1/\alpha}}$ ([4] p.281). When X_t is spectrally negative, the formula (8) was obtained recently by T. Simon([17]).

Proof. We note that if $g(0) = 1$ then

$$\int_0^\infty \frac{1}{x+t} f(x)dx = g(t) \quad \Longrightarrow \quad \int_0^\infty \frac{1}{x+t}\frac{f(x)}{x}dx = \frac{1}{t} - \frac{g(t)}{t}. \tag{9}$$

Indeed

$$1 = \int_0^\infty \frac{x+t}{x+t}\frac{f(x)}{x}dx = g(t) + \int_0^\infty \frac{t}{x+t}\frac{f(x)}{x}dx$$

From [14] we know that

$$\int_0^\infty e^{-\theta y}\mathbb{E}^y e^{-\eta\tau}\,dy = \frac{1}{\theta} - \frac{\widehat{\kappa}(\eta,0)}{\theta\widehat{\kappa}(\eta,\theta)}, \tag{10}$$

where $\tau = \tau_{(0,\infty)}$ and $\eta, \theta > 0$. Putting $\eta = 1$ and applying (9) to (7) we get

$$\begin{aligned}
&\int_0^\infty e^{-\theta y}\mathbb{E}^y e^{-\tau}\,dy \\
=&\frac{\sin((1-\rho)\alpha\pi)}{\pi}\int_0^\infty \frac{1}{x+\theta}\frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}\,dx \\
=&\frac{\sin((1-\rho)\alpha\pi)}{\pi}\int_0^\infty\int_0^\infty e^{-y(x+\theta)}\frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}\,dy\,dx \\
=&\int_0^\infty e^{-\theta y}\left(\frac{\sin((1-\rho)\alpha\pi)}{\pi}\int_0^\infty e^{-yx}\frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}\,dx\right)dy.
\end{aligned}$$

Therefore

$$\mathbb{E}^y e^{-\tau} = \frac{\sin((1-\rho)\alpha\pi)}{\pi}\int_0^\infty e^{-yx}\frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}\,dx$$

and assertion of the theorem follows from scaling property $\mathbb{E}^y e^{-\tau} = \mathbb{E}^1 e^{-y^\alpha\tau}$ of stable processes. $\square$

For $\alpha \geq 1$ we immediately obtain from Theorem 3 a formula for the density $h = h_{\alpha,\rho}$ of τ under $\mathbb{P}^1$.

Corollary 4. *Let $\alpha \in (1,2)$. The density of $\tau = \tau_{(0,\infty)}$ under $\mathbb{P}^1$ is given by*

$$h(s) = \frac{\sin((1-\rho)\alpha\pi)}{\pi} \int_0^\infty \eta_{1/\alpha}(x,s) \frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha} + 2x^\alpha \cos((1-\rho)\alpha\pi) + 1}\, dx, \quad (11)$$

where $\eta_\gamma(t,x)$ is the transition density of γ-stable subordinator.

Proof. Since $\int_0^\infty e^{-xs}\eta_\gamma(t,x)dx = e^{-ts^\gamma}$ we obtain by Theorem 3

$$\begin{aligned}
&\int_0^\infty e^{-st} h_\alpha(s)\, ds \\
&= \frac{\sin((1-\rho)\alpha\pi)}{\pi} \int_0^\infty \int_0^\infty e^{-st}\eta_{1/\alpha}(x,s) \frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha} + 2x^\alpha \cos((1-\rho)\alpha\pi) + 1}\, ds\, dx \\
&= \int_0^\infty e^{-st} \frac{\sin((1-\rho)\alpha\pi)}{\pi} \int_0^\infty \eta_{1/\alpha}(x,s) \frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha} + 2x^\alpha \cos((1-\rho)\alpha\pi) + 1}\, dx\, ds
\end{aligned}$$

and the assertion follows. □

Corollary 5. *If X_t is a Cauchy process on $\mathbb{R}$ ($\alpha = 1, \rho = 1/2$) then the density of the exit time $\tau = \tau_{(0,\infty)}$ under $\mathbb{P}^1$ is given by*

$$h(x) = \frac{1}{\pi} \frac{\kappa(1,x)}{x^2+1}, \quad x \geq 0.$$

Remark. The above formula for $h_{1,1/2}$ was obtained previously by Darling in [5] (see also [11, (7.13)]).

For $\alpha \neq 1$, Theorem 3 gives interesting multiplicative convolution relations verified by τ. We present them in the following subsection.

3.1 Interpretation in terms of multiplicative convolutions

For a given strictly α-stable process X_t with $\rho \in [0,1) \setminus \{1-1/\alpha\}$ we define the following function on $\mathbb{R}^+$

$$m_{\alpha,\rho}(x) = \frac{\sin((1-\rho)\alpha\pi)}{\pi\alpha} \frac{\kappa(1,x^{1/\alpha})}{x^2 + 2x\cos((1-\rho)\alpha\pi) + 1}, \quad x \geq 0.$$

Observe that $m_{\alpha,\rho}(x)$ is a probability density on $\mathbb{R}^+$. This follows from the formula

$$\frac{\sin((1-\rho)\alpha\pi)}{\pi}\int_0^\infty \frac{x^{\alpha-1}\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}\,dx = 1$$

obtained from (8) when $t\to 0$ and by a change of variables $x = y^{1/\alpha}$.

Denote by $M_{\alpha,\rho}$ a positive random variable with density $m_{\alpha,\rho}$. The variable $M_{\alpha,1/\alpha}$, $1<\alpha<2$ appeared for the first time in [17] for the special case of a completely asymmetric α-stable process, $1<\alpha<2$ (in our context a spectrally negative process), when $\kappa(1,x^{1/\alpha}) = 1 + x^{1/\alpha}$.

Let $\gamma\in(0,1)$ and $\eta_\gamma(t,x)$ be the transition density of γ-stable subordinator. Denote by $N(\gamma)$ a random variable with the density $\eta_\gamma(1,x)$, i.e. $\mathbb{E}\exp(-xN(\gamma)) = e^{-x^\gamma}$, $x\ge 0$.

Recall that if Y and Z are independent random variables on $(0,\infty)$ with densities f and g respectively, then the multiplicative convolution $Y\times Z^p$ is a random variable with the density

$$\mathbb{P}[Y\times Z^p\in dt] = \int_0^\infty f(\frac{t}{u^p})g(u)\frac{du}{u^p}. \tag{12}$$

Corollary 6. *(i) Let $1<\alpha<2$. Suppose that random variables $M_{\alpha,\rho}$ and $N(1/\alpha)$ are independent and that $X_0 = 1$. We have*

$$\tau \stackrel{d}{=} M_{\alpha,\rho}\times N(1/\alpha),$$

(ii) Let $0<\alpha<1$. Suppose that random variables τ and $N(\alpha)$ are independent. We have

$$\tau\times N(\alpha)^\alpha \stackrel{d}{=} M_{\alpha,\rho}.$$

Proof. Part (i) follows immediately from Corollary 4. In (11) we substitute $x^\alpha = u$ and use the scaling property $\eta_{1/\alpha}(u^{1/\alpha},s) = u^{-1}\eta_{1/\alpha}(1,su^{-1})$.

In order to prove (ii), we use $e^{-ty} = \mathbb{E}[-(ty)^{1/\alpha}N(\alpha)]$ in the left-hand side of (8), we apply Fubini and change variables $x = y^{1/\alpha}u$. By unicity of the Laplace transform we get

$$\int_0^\infty h_\alpha(\frac{x^\alpha}{u^\alpha})\eta_\alpha(u)\frac{du}{u^\alpha} = \frac{\sin((1-\rho)\alpha\pi)}{\pi\alpha}\frac{\kappa(1,x)}{x^{2\alpha}+2x^\alpha\cos((1-\rho)\alpha\pi)+1}$$

Replacing x^α by x in the last formula and using (12) ends the proof of (ii). $\square$

3.2 Application for the Doney's class $\hat{C}_{1,1}$

Let $\alpha \in [\frac{1}{2}, 1)$ and $1 - \rho = 1/\alpha - 1$, i.e. we consider a process dual to a process from the class $C_{1,1}$ from R. Doney's article [7]. We denote the class of such processes by $\hat{C}_{1,1}$. In this case we have by [7] or [9]

$$\kappa(1, x) = \frac{x^{2\alpha} - 2x^\alpha \cos\alpha\pi + 1}{1 + x}.$$

We recall the definition of the confluent hypergeometric functions ${}_1F_1$ and U (see [1])

$$\begin{aligned}
&{}_1F_1(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \quad z \in \mathbb{C}, \\
&U(a, b, z) \\
&= \frac{\pi}{\sin \pi b} \left(\frac{{}_1F_1(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{{}_1F_1(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)} \right), \quad z \in \mathbb{C} \setminus \mathbb{R}_-,
\end{aligned}$$

where $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$ is Pochhammer symbol. Using Theorem 3 we get the following formulas for Laplace and Stieltjes transforms of τ

Proposition 7. *If $\alpha \in [\frac{1}{2}, 1)$ and $X_t \in \hat{C}_{1,1}$ then*

$$(i) \quad \mathbb{E}^1 e^{-\tau t} = \frac{\sin(\alpha\pi)}{\pi} \Gamma(\alpha)\Gamma(1 - \alpha, t^{1/\alpha}) e^{t^{1/\alpha}}, \tag{13}$$

$$(ii) \quad \mathbb{E}^1 \frac{1}{x + \tau} = \int_0^\infty e^{-u} \left(\frac{e^{u^{1/\alpha} x^{-1/\alpha}}}{x} - \frac{u^{1/\alpha - 1}}{x^{1/\alpha}} \frac{{}_1F_1(1, 2 - \alpha, u^{1/\alpha} x^{-1/\alpha})}{\Gamma(2 - b)} \right) du, \tag{14}$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t}\, dt$ is the incomplete Gamma function.

Proof. To prove (i) we use simple transformations of integrals

$$
\begin{aligned}
\mathbb{E}^1 e^{-\tau t} &= \frac{\sin(\alpha(1/\alpha-1)\pi)}{\pi}\int_0^\infty \frac{e^{-t^{1/\alpha}x}x^{\alpha-1}(x^{2\alpha}-2x^\alpha\cos\alpha\pi+1)}{(1+x)(x^{2\alpha}+2x^\alpha\cos(\alpha(1/\alpha-1)\pi)+1)}\,dx \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{e^{-t^{1/\alpha}x}x^{\alpha-1}}{1+x}\,dx \qquad (15)\\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty\int_0^\infty e^{-t^{1/\alpha}x}e^{-s(x+1)}x^{\alpha-1}\,ds\,dx \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-s}\frac{\Gamma(\alpha)}{(t^{1/\alpha}+s)^\alpha}\,ds = \frac{\sin(\alpha\pi)\Gamma(\alpha)}{\pi}\int_{t^{1/\alpha}}^\infty e^{t^{1/\alpha}-r}\frac{1}{r^\alpha}\,dr \\
&= \frac{\sin(\alpha\pi)}{\pi}\Gamma(\alpha)\Gamma(1-\alpha,t^{1/\alpha})e^{t^{1/\alpha}}.
\end{aligned}
$$

To prove (ii) we use the following integral representation of U (see [1])

$$
U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt, \qquad \operatorname{Re} z > 0.
$$

Applying this to (15) we get

$$
\begin{aligned}
\mathbb{E}^1\frac{1}{x+\tau} &= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty\int_0^\infty e^{-xt}e^{-t^{1/\alpha}s}\frac{s^{\alpha-1}}{1+s}\,ds\,dt \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty\int_0^\infty e^{-u}e^{-u^{1/\alpha}x^{-1/\alpha}s}\frac{s^{\alpha-1}}{x(1+s)}\,ds\,du \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-u}\Gamma(\alpha)\frac{U(\alpha,\alpha,u^{1/\alpha}x^{-1/\alpha})}{x}\,du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}^1\frac{1}{x+\tau} &= \int_0^\infty \frac{e^{-u}}{x}\left({}_1F_1(\alpha,\alpha,u^{1/\alpha}x^{-1/\alpha}) - \frac{u^{1/\alpha-1}}{x^{1/\alpha-1}}\frac{{}_1F_1(1,2-\alpha,u^{1/\alpha}x^{-1/\alpha})}{\Gamma(2-b)}\right)du \\
&= \int_0^\infty e^{-u}\left(\frac{e^{u^{1/\alpha}x^{-1/\alpha}}}{x} - \frac{u^{1/\alpha-1}}{x^{1/\alpha}}\frac{{}_1F_1(1,2-\alpha,u^{1/\alpha}x^{-1/\alpha})}{\Gamma(2-b)}\right)du.
\end{aligned}
$$

□

It is possible to invert Stieltjes transform in (14) and the resulting density of τ is given by a series either in x or in $1/x$ (cf. [12]). For $\alpha = 2/3$ the process X_t is symmetric and the density of τ has a nice integral representation involving hypergeometric function

$$ {}_1F_2(a;b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k (c)_k k!}, \qquad z \in \mathbb{C}. $$

Corollary 8. *Let X_t be a symmetric $\frac{2}{3}$-stable process on $\mathbb{R}$ with $X_0 = 1$. Then the density of $\tau = \tau_{(0,\infty)}$ is given by the formula*

$$ \mathbb{P}^1(\tau \in dx) = \frac{1}{\pi} \int_0^\infty e^{-tx} \left(\sin(t^{3/2}) + \frac{t^{1/2}{}_1F_2(1;2/3,7/6;-t^3/4)}{\Gamma(4/3)} \right) dt \quad (16) $$

and its Laplace transform is

$$ \mathbb{E}^1 e^{-t\tau} = \frac{\sqrt{3}}{2\pi} \Gamma(2/3)\Gamma(1/3, t^{3/2}) e^{t^{3/2}}, \qquad t > 0. $$

Proof. The second part is a direct consequence of (13). By (14)

$$ \mathbb{E}^1 \frac{1}{x+\tau} = \int_0^\infty e^{-u} \left(\frac{e^{u^{3/2}x^{-3/2}}}{x} - \frac{u^{1/2}}{x^{3/2}} \frac{{}_1F_1(1,4/3,u^{3/2}x^{-3/2})}{\Gamma(4/3)} \right) du. $$

Inverting Stieltjes transform we get

$$ \begin{aligned} &\mathbb{P}^1(\tau \in dx) \\ &= -\frac{1}{\pi} \operatorname{Im} \int_0^\infty e^{-u} \left(\frac{e^{u^{3/2}x^{-3/2}e^{-3i\pi/2}}}{-x} - \frac{u^{1/2}}{x^{3/2}e^{3i\pi/2}} \frac{{}_1F_1(1,4/3,u^{3/2}x^{-3/2}e^{-3i\pi/2})}{\Gamma(4/3)} \right) du \\ &= -\frac{1}{\pi} \operatorname{Im} \int_0^\infty e^{-u} \left(\frac{e^{u^{3/2}x^{-3/2}i}}{-x} - i\frac{u^{1/2}}{x^{3/2}} \frac{{}_1F_1(1,4/3,u^{3/2}x^{-3/2}i)}{\Gamma(4/3)} \right) du. \end{aligned} $$

Since

$$ \begin{aligned} \operatorname{Im} i\, {}_1F_1(1,4/3,yi) &= \operatorname{Re} \sum_{k=0}^{\infty} \frac{(1)_k (iy)^k}{k!(4/3)_k} = \sum_{k=0}^{\infty} \frac{(-y^2)^k}{(4/3)_{2k}} \\ &= \sum_{k=0}^{\infty} \frac{(1)_k (-y^2)^k}{k!(4/6)_k (7/6)_k 2^{2k}} = {}_1F_2(1;2/3,7/6;-y^2/4) \end{aligned} $$

we finally get

$$
\begin{aligned}
&\mathbb{P}^1(\tau \in dx)\\
&= \frac{1}{\pi}\int_0^\infty e^{-u}\left(\frac{\sin(u^{3/2}x^{-3/2})}{x} + \frac{u^{1/2}}{x^{3/2}}\frac{{}_1F_2(1;2/3,7/6;-u^3x^{-3}/4)}{\Gamma(4/3)}\right)du\\
&= \frac{1}{\pi}\int_0^\infty e^{-tx}\left(\sin(t^{3/2}) + \frac{t^{1/2}{}_1F_2(1;2/3,7/6;-t^3/4)}{\Gamma(4/3)}\right)dt.
\end{aligned}
$$

□

Remark. It is possible to obtain the formula (16) from the results of M. Kwaśnicki [13]. However, in order to do this, one has to make several non-elementary transformations of integrals and our approach seems simpler than using [13].

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